LOOKING AT THE TSALLIS ENTROPY IN THE EYE

G. Baris Bagci[1,*], Thomas Oikonomou[2]

[1]Department of Materials Science and Nanotechnology Engineering, TOBB University of Economics and Technology, Turkey
[2]Department of Physics, School of Science and Technology, Nazarbayev University, Kazakhstan
[*]gbb0002@hotmail.com

ABSTRACT

In Information theory, Shannon [1] first introduced the concept of information gain or surprise $I(p_i)$ as

$$I(p_i) = \ln(1/p_i) \tag{1}$$

where p_i denotes the probability of the ith event. The Shannon entropy S just happens to be the linear average of this fundamental concept. Apart from some other properties of the logarithmic function, Shannon has particularly chosen this expression, since it allows events with zero probability to have the maximum surprise i.e.

$$\lim_{p_i \to 0} I(p_i) = \lim_{p_i \to 0} \ln(1/p_i) = +\infty. \tag{2}$$

One, however, does not need to study the information theory to be able to appreciate the importance of the above limit concerning the information gain or surprise. Imagine now that you have the canonical distribution function $p_i = \exp(-\beta\varepsilon_i)$ apart from the normalization factor. When this is the case, the limit in Eq. (2) uniquely ensures the validity of the third law of thermodynamics since

$$\lim_{p_i \to 0} \ln(1/p_i) = +\infty \quad \text{if and only if} \quad \beta \longrightarrow +\infty. \tag{3}$$

In other words, this particular limiting behaviour of the information-theoretical gain or surprise ensures the validity of the third law of thermodynamics in a statistical mechanical context.

The non-additive Tsallis entropy [2] also relies on the definition of the information gain or surprise. The sole change is that one uses the q-deformed logarithms to obtain the Tsallis entropy instead of the usual logarithm for the Shannon entropy. This q-deformed logarithm usually reads

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}. \tag{4}$$

It is then easy to see that the Tsallis entropy is the linear average of the surprise written in terms of the q-deformed logarithm i.e.

$$S_q = \langle \ln_q(1/p_i) \rangle = \frac{\sum_i p_i^q - 1}{1-q} \tag{5}$$

just as the Shannon entropy reads

$$S = \langle \ln(1/p_i) \rangle = \sum_i p_i \ln(1/p_i). \tag{6}$$

However, the q-deformation in Eq. (5) partially preserves the limiting behaviour of the ordinary definition of the surprise in Eq. (2) since

$$\lim_{p_i \to 0} \ln_q(1/p_i) = +\infty \tag{7}$$

only for $q \in (0,1]$ [3]. In fact, one can see that

$$\lim_{p_i \to 0} \ln_q(1/p_i) = \frac{1}{q-1} \tag{8}$$

for $q \in [1,2)$ [3].

As one might already begin to suspect, the Tsallis entropy then satisfies the third law of thermodynamics only for $q \in (0,1]$ [3]. To show this explicitly, one can borrow the expression obtained from Bento *et al.* [4], which allows one to check the third law by referring to the micro-probabilities p_i. This expression reads

$$\beta_n = \frac{\partial S}{\partial p_n} \left(\frac{\partial U}{\partial p_n} \right)^{-1} \tag{9}$$

where $n = 1, ..., N$ excluding the ground state and β_n is the inverse temperature due to the nth energy level.

The third law test checks whether this inverse temperature β_n approaches to infinity when $\{p_n\} \to 0$ as $p_0 = 1$, implying that only the lowest energy level is occupied while all the other levels are empty. Using the ordinary internal energy definition $U = \sum_i p_i E_i$, Eq. (9) together with Eq. (5) yield

$$\beta_n = -q \lim_{(p_n,p_0)\to(0,1)} \frac{\ln_{2-q}(p_n)-\ln_{2-q}(p_0)}{(E_n-E_0)} = q \lim_{p_n\to 0}\frac{\ln_q(1/p_n)}{(E_n-E_0)} = +\infty \tag{10}$$

which is valid only for the interval $q \in (0,1]$ as expected, since the numerator on the far right side of the equation above is nothing but the term $\lim_{p_n\to 0}\ln_q(\frac{1}{p_n})$ in Eq. (7).

Then, the questions remains as to whether we have an entropy definition for the super-additive region $q>1$. In order to shed light on this issue, we observe the Shannon entropy given by Eq. (6) and note that it can also be written as $S = \langle \ln(1/p_i)\rangle = -\sum_i p_i \ln(p_i)$ using the relation

$$\ln(p_i)+\ln(1/p_i) = 0. \tag{11}$$

A similar equation in the case of q-deformed logarithms read

$$\ln_q(p_i)+\ln_{2-q}(1/p_i) = 0. \tag{12}$$

Using the above relation, we obtain

$$S_q = \langle \ln_{2-q}(1/p_i)\rangle = \langle -\ln_q(p_i)\rangle = \frac{\sum_i p_i^{2-q}-1}{q-1}. \tag{13}$$

Note that we now have

$$\lim_{p_i\to 0}\ln_{2-q}(1/p_i) = +\infty \tag{14}$$

only in the interval $q\in[1,2)$. Therefore, one expects that the Tsallis entropy in Eq. (13) should conform to the third law of thermodynamics for the interval $q\in[1,2)$. In fact, a quick calculation using Eqs. (9) and (13) yields

$$\beta_n = (q-2)\lim_{(p_n,p_0)\to(0,1)}\frac{\ln_q(p_n)-\ln_q(p_0)}{(E_n-E_0)} = (2-q)\lim_{p_n\to 0}\frac{\ln_{2-q}(1/p_n)}{(E_n-E_0)} = +\infty \tag{15}$$

which is valid only for the interval $q\in[1,2)$. Note that the numerator on the far right side of the equation above is exactly $\lim_{p_n\to 0}\ln_{2-q}(1/p_n)$ in Eq. (14). Moreover, we have again made use of the ordinary internal energy constraint $U=\sum_i p_i E_i$.

To sum up the progress so far, we have seen that the non-additive Tsallis entropy has two distinct expressions for two distinct intervals [3]. One has the usual expression

$$S_q = \langle \ln_q(1/p_i)\rangle = \frac{\sum_i p_i^q - 1}{1-q} \tag{16}$$

for the interval $q\in(0,1]$ [3] and one has

$$S_q = \langle \ln_{2-q}(1/p_i)\rangle = \langle -\ln_q(p_i)\rangle = \frac{\sum_i p_i^{2-q}-1}{q-1} \tag{17}$$

for the interval $q\in[1,2)$ [3].

The careful reader will notice that we did **NOT** use the escort definition of the internal energy constraint i.e. $U=\frac{\sum_i p_i^q E_i}{\sum_i p_i^q}$ at all. The simple reason for this is that we did not need to. In fact, if we maximize the entropy expressions in Eqs. (16) and (17) subject to the ordinary definition $U=\sum_i p_i E_i$, we obtain the equilibrium distributions

$$1/p_i = \exp_q\left(\frac{1+\alpha+\beta E_i}{q}\right) \tag{18}$$

which is valid only for $q\in(0,1]$, and

$$1/p_i = \exp_{2-q}\left(\frac{1+\alpha+\beta E_i}{2-q}\right) \tag{19}$$

valid only for $q\in[1,2)$, respectively. The expression $\exp_q(x)$ denotes the q-exponential defined as $\exp_q(x) = [1+(1-q)x]^{\frac{1}{1-q}}$.

The distribution in Eq. (19) is what is called the escort distribution in the literature and it is a general practice to obtain it by using the entropy in Eq. (16) together with the constraint $U=\frac{\sum_i p_i^q E_i}{\sum_i p_i^q}$. However, note that it is solely obtained from the ordinary internal energy definition $U=\sum_i p_i E_i$ here.

The expressions (16)-(19) form the complete Tsallis entropy framework. By adopting the complete Tsallis expressions, we have a theory which conforms to the third law, yielding the correct surprise limits as an information-theoretical entropy and is Lesche stable. The alternative route includes the *ad-hoc* use of the $(2-q)$ transformation, a difficult-to-justify escort internal energy expression $U=\frac{\sum_i p_i^q E_i}{\sum_i p_i^q}$ and breaking the linear averaging scheme inherent in the core definition of the Tsallis entropy (see Eqs. (16) and (17)).

Before proceeding further, it is worth remarking that the preceding q-values are in full agreement with the ones obtained within the "complete Tsallis entropy" [5] formalism, which precedes even the 3rd law of thermodynamics regarding the core structure of the q-deformed logarithms. Note that one has

$$\ln_q(x)+\ln_q(1/x) = \frac{x^{1-q}+x^{q-1}-2}{1-q} \neq 0 \tag{20}$$

where $x>0$ and Eq. (20) is the incompleteness property of the q-logarithm. This simple relation reveals that the q-logarithm cannot yield all values in the set of real numbers contrary to the natural logarithm. On top of it, this restriction on the set of values deformed logarithm can take is generally q-dependent just because of the incompleteness of the q-deformed logarithms. In other words, the results of the non-extensive theory may be the ones chosen from those q values dictated by the narrower range of values limited by Eq. (20), instead of being fully imposed by the physical system i.e. the Nature (please see Ref. [5] for more rigorous treatment of this issue).

Lastly, even though one adopts the aforementioned complete Tsallis expressions, we point out that one should also consistently maximize the entropies. In this regard, we provide an explicit historical example from the first paper on the Tsallis entropy in the context of generalized thermostatistics [6]. In this example, one maximizes the functional Φ as

$$\Phi(\{p_i\},\alpha,\beta) = \frac{\sum_i p_i^q - 1}{1-q} + \alpha\left[\sum_{i=1}^{n} p_i - 1\right] + \alpha\beta(1-q)\left[\sum_{i=1}^{n} p_i \varepsilon_i - U\right] \tag{21}$$

and obtains a q-exponential distribution. What is important to observe is that, as always, the canonical distribution should be recovered in the limit $q \to 1$. In fact, this functional should yield the usual maximization procedure yielding the canonical distribution if the Tsallis entropy is indeed to be a generalization of the Shannon entropy.

If one considers the limit $q \to 1$ in Eq. (21), one should also have $\alpha(1-q) \to 1$ to recover the usual maximization functional Φ for the canonical distribution. However, as $q \to 1$, one has $\alpha(1-q) \to 0$, the only other option being a diverging α. In other words, in the limit $q \to 1$, the functional Φ in Eq. (21) becomes

$$\lim_{q\to 1} \Phi(\{p_i\},\alpha,\beta) = -\sum_i p_i \ln p_i + \alpha\left[\sum_{i=1}^{n} p_i - 1\right] \tag{22}$$

which is the functional for the micro-canonical distribution. In other words, one obtains the canonical distribution from the well-known Tsallis limit $q \to 1$ by using the micro-canonical maximization procedure [7]. To put it even more strangely, the Tsallis theory implies that we should recover the ordinary canonical distribution $p_i = \exp(-\beta E_i)$ only by assuming the normalization constraint $\sum_{i=1}^{n} p_i = 1$ [7].

What went wrong in the example above then? A brief moment of thinking is enough to convince one that the issue above stems from the process of coupling the Lagrange multipliers as $\alpha\beta(1-q)$. The lesson to be learned from the above example is then not to couple the Lagrange multipliers!

One might then be tempted to write the functional Φ in Eq. (21) without coupling and in terms of the escort distribution as

$$\Phi(\{p_i\},\alpha,\beta) = \frac{\sum_i p_i^q - 1}{1-q} + \alpha\left[\sum_{i=1}^{n} p_i - 1\right] + \beta\left[\frac{\sum_i p_i^q E_i}{\sum_i p_i^q} - U\right] \tag{23}$$

Similarly, if we take the canonical limit $q \to 1$ in Eq. (23), one has

$$\lim_{q\to 1} \Phi(\{p_i\},\alpha,\beta) = -\sum_i p_i \ln p_i + \alpha\left[\sum_{i=1}^{n} p_i - 1\right] + \beta\left[\frac{\sum_i p_i E_i}{\sum_i p_i} - U\right] \tag{24}$$

The equation above shows that it is still not the ordinary canonical functional to be maximized. The crucial point to understand is that one now has the term $\left[\frac{\sum_i p_i E_i}{\sum_i p_i} - U\right]$ in Eq. (24) instead of what should have been, namely, $[\sum_i p_i E_i - U]$. Note also that one cannot equate $\left[\frac{\sum_i p_i E_i}{\sum_i p_i} - U\right]$ to the term $[\sum_i p_i E_i - U]$, since the normalization $\sum_{i=1}^{n} p_i = 1$ is not ensured yet i.e. before the maximization procedure is carried out. The presence of this additional multiplicative term $\sum_{i=1}^{n} p_i = 1$ in the functional before maximization has dire consequences such as violating the thermodynamic structure of the ordinary thermodynamics [8].

REFERENCES

[1] Claude E. Shannon, A Mathematical Theory of Communication, *Bell System Technical Journal,* vol. 27 (3), pp. 379-423, 1948.

[2] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics*, Springer, New York, 2009.

[3] G. B. Bagci and Thomas Oikonomou, The Validity of the Third Law of Thermodynamics For the Tsallis Entropy, *Physical Review E,* vol. 93, pp. 022112 (1)-022112 (4), 2016.

[4] E. P. Bento, G. M. Viswanathan, M. G. E. da Luz and R. Silva, Third Law of Thermodynamics as a Key Test of Generalized Entropies, *Physical Review E,* vol. 91, pp. 022105 (1)-022105 (7), 2015.

[5] Thomas Oikonomou and G. Baris Bagci, The Maximization of Tsallis Entropy with Complete Deformed Functions and the Problem of Constraints, *Physics Letters A,* vol. 374, pp. 2225-2229, 2010.

[6] C. Tsallis, Possible Generalization of Boltzmann-Gibbs Statistics , *J. Stat. Phys.,* vol. 52, pp. 479-487, 1988.

[7] Thomas Oikonomou and G. B. Bagci, Misusing the Entropy Maximization in the Jungle of Generalized Entropies, *Physics Letters A,* vol. 381, pp. 207-211, 2017.

[8] Thomas Oikonomou and G. B. Bagci, submitted.