RÉNYI ENTROPY RATE UNDER LINDBLAD EQUATION

Sumiyoshi Abe[1,2,3,*]

[1]Physics Division, Faculty of Information Science and Engineering, Huaqiao University, Xiamen 361021, China
[2]Department of Physical Engineering, Mie University, Mie 514-8507, Japan
[3]Institute of Physics, Kazan Federal University, Kazan 420008, Russia
[*]suabe@sf6.so-net.ne.jp

ABSTRACT

The quantum-mechanical version of the Rényi entropy [1] is defined by

$$S_\alpha = \frac{1}{1-\alpha} \ln \operatorname{tr} \rho^\alpha . \tag{1}$$

Here, ρ is a normalized density matrix that is self-adjoint and positive semidefinite. α is the positive entropic index. The von Neumann entropy, $S = -\operatorname{tr}(\rho \ln \rho)$, is realized in the limit $\alpha \to 1$. As well known, in the case when the system is in a pure state, the density matrix is idempotent, and accordingly the Rényi entropy identically vanishes. Therefore, the nonvanishing Rényi entropy implies that the system is in a mixed state. In other words, the system in such a case is actually a subsystem surrounded by the environment. The total system composed of the subsystem and environment can be regarded as an isolated one, a state of which may be pure. Then, a density matrix describing a quantum state of the subsystem is obtained by the partial trace over the environment. In order for the subsystem to be in a mixed state, the existence of entanglement between the subsystem and environment before the partial trace is essential.

In recent years, the Rényi entropy in Eq. (1) has repeatedly been discussed for characterizing quantum critical states and entanglement [2-4]. A reason behind this may be due to the fact that the index α makes it possible to "dictate" the nature of entanglement contained in the subsystem.

To generalize the discussion to include dynamical evolution, it is necessary to chose a master equation for a density matrix. This is a highly nontrivial issue, in general. The situation, however, becomes radically simplified if Markovianity of the subdynamics is assumed. It is known, in this case, that the linear master equation preserving positive semidefiniteness of a density matrix is uniquely given by the Lindblad equation [5,6], which generates a dynamical semigroup and has the following form:

$$i \frac{\partial \rho}{\partial t} = [H, \rho] - \frac{i}{2} \sum_n c_n \left(L_n^\dagger L_n \rho + \rho L_n^\dagger L_n - 2 L_n \rho L_n^\dagger \right) . \tag{2}$$

In this equation, H is the "sub-Hamiltonian", i.e., the Hamiltonian of the subsystem (later, we shall make a comment on this quantity). L_n's are the Lindbladian operators that are responsible with the nonunitarity of the subdynamics. c_n's are nonnegative c-number coefficients. Here and hereafter, $\hbar$ is set equal to unity.

The subsequent discussion has two parts. In the first part, we report a recent result given in Ref. [7] about the lower bound on the Rényi entropy rate under the Lindblad equation in Eq. (2). Then, in the second part, we make several critical comments on maximum entropy methods with generalized entropies .

The main result about the time derivative of the Rényi entropy combined with the Lindblad equation is summarized as follows:

$$\frac{dS_\alpha}{dt} = \sum_n c_n \Gamma_n , \tag{3}$$

$$\Gamma_n > \left\langle \left[L_n^\dagger, L_n \right] \right\rangle_\alpha . \tag{4}$$

The symbol appearing on the right-hand side of Eq. (4) is referred to as the *escort average* and is defined by

$$\left\langle A \right\rangle_\alpha = \frac{\operatorname{tr}\left(A \rho^\alpha \right)}{\operatorname{tr} \rho^\alpha} . \tag{5}$$

An elementary proof of this can be found in Ref. [7].

From the result, it immediately follows that the Rényi entropy increases in time if the Lindbladian operators are *normal*, i.e., $[L_n^\dagger, L_n] = 0$ (self-adjointness is a trivial case of normality). This generalizes the theorem previously proven for the von Neumann entropy [8] (see also Ref. [9]).

Now, on the above-mentioned concepts, quantities and results, we wish to make the following four comments, which seem to be useful for the reader interested in maximum generalized-entropy methods.

The first comment is on the fact that the result presented above has nothing to do with the second law of thermodynamics, since it is not possible to construct (generalized) statistical mechanics based on generalized entropies such as the Rényi entropy. Maximum entropy methods with generalized entropies contain unwarranted biases [10,11]. (Although the Rényi entropy is additive unlike in the title of Ref. [10], the same discussion applies.). This point is related to the so-called Shore-Johnson axioms [12-14], which consist of the following five ones: (I) Uniqueness; If the same problem is solved twice, then the same answer is expected to result both times, (II) Invariance; The same answer is expected when the same problem is solved in two different coordinate systems, in which the posteriors in the two systems should be related by the coordinate transformation, (III) System independence; It should not matter whether one accounts for independent information about independent systems separately in terms of their marginal distributions or in terms of the joint distribution, (IV) Subset independence; It should not matter whether one treats independent subsets of the states of the systems in terms of their separate conditional distributions or in terms of the joint distribution, and (V) Expansibility; In the absence of new information, the prior, i.e., the reference distribution, should not be changed. The requirement of these axioms avoids introducing unwarranted biases into the schemes for inference of a distribution function based on entropy maximization and cross-entropy minimization. Care has to be taken for (III): It never means that correlations are excluded. The maximum Rényi entropy method and related ones have also a severe difficulty even at the level of the normalizability condition on a maximum-entropy state if a many-body system is considered [15-17].

The second comment is on the problems regarding uniform continuities of the Rényi entropy and escort average as operator functionals [18-23]. The issue is concerned with the concept that is referred to in the literature as the Lesche stability. Physically, this means as follows. Consider a certain quantity that depends functionally on a density matrix and can experimentally be measured. Then, it is natural to require that for *any* two density matrices, which are slightly different from each other, the corresponding values of the quantity should also be slightly different. To see this more explicitly, let us diagonalize the density matrix by employing the orthonormal basis $\{|u_i\rangle\}_{i=1,2,\ldots,W}$: $\rho = \sum_{i=1}^{W} p_i |u_i\rangle\langle u_i|$, where W is the number of states (that is enormously large in the thermodynamic limit) and p_i's are nonnegative eigenvalues of the density matrix satisfying the normalization condition $\sum_{i=1}^{W} p_i = 1$. In this representation, the Rényi entropy in Eq. (1) takes the "classical form": $S_\alpha = [1/(1-\alpha)] \ln \sum_{i=1}^{W} (p_i)^\alpha$. Similarly, the escort average in Eq. (5) is written as $\langle Q \rangle_\alpha = \sum_{i=1}^{W} Q_i P_i^{(\alpha)}$, where $P_i^{(\alpha)} = (p_i)^\alpha / \sum_{j=1}^{W} (p_j)^\alpha$ and $Q_i = \langle u_i | Q | u_i \rangle$. Both of these are the functionals of p_i's. Let $G = G[p]$ be a physical quantity as a functional of p_i's. Take p'_i close to p_i , that is, the l^1 – norm $\|p - p'\|_1 = \sum_{i=1}^{W} |p_i - p'_i|$ is small. Then, the Lesche stability is expressed by the following predicate: $(\forall \varepsilon > 0)$ $(\exists \delta > 0)$ $\left(\|p - p'\|_1 < \delta \Rightarrow |G[p] - G[p']| / |G|_{\max} < \varepsilon\right)$ for any value of W, where $|G|_{\max} = \max_p |G[p]|$. Neither the Rényi entropy nor the escort average is Lesche-stable [18-23]. This result reveals a fatal flaw in the maximum entropy method with generalized entropies. The Shore-Johnson axioms require the use of escort average [24], whereas the escort average suffers from a series of conceptual difficulties [23] (see also Ref. [25], although amendment is not possible in spite of its title) in addition to the violation of the Lesche stability condition.

The third comment is on the existence of the sub-Hamiltonian. If the total system is strongly entangled, then it will not be possible to define the sub-Hamiltonian of the subsystem. This has a thermodynamic analogy. The internal energy is a well-defined quantity if the correlation between the subsystem and environment is negligibly weak.

The last comment is on Markovianity. If entanglement is very strong (e.g., at or near quantum criticality), then the third comment will again be relevant and, at the same time, the temporal locality of the dynamics will also be questioned. Transitions between states are induced by fluctuations that may be spread both spatially and temporally. Accordingly, the subdynamics needs its non-Markovian formulation [26].

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